

# ISOMETRIC ISOMORPHISMS OF THE ANNIHILATOR OF $C_0(G)$ IN $LUC(G)^*$

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**ABSTRACT.** Let  $LUC(G)$  denote the  $C^*$ -algebra of left uniformly continuous functions with the uniform norm and let  $C_0(G)^\perp$  denote the annihilator of  $C_0(G)$  in  $LUC(G)^*$ . In this article, among other results, we show that if  $G$  is a locally compact group and  $H$  is a discrete group then whenever there exists a weak-star continuous isometric isomorphism between  $C_0(G)^\perp$  and  $C_0(H)^\perp$ ,  $G$  is isomorphic to  $H$  as a topological group. In particular, when  $H$  is discrete  $C_0(H)^\perp$  determines  $H$  within the class of locally compact topological groups.

## 1. INTRODUCTION AND PRELIMINARIES

In [Wen51] Wendel showed that two locally compact groups are isomorphic if their group algebras are isometrically isomorphic. Similar results for different algebras have since been proved. Johnson [Joh64] proved that the same conclusion holds when the group algebra is replaced by a measure algebra. Lau and McKennon in [LM80] generalized Johnson's result. They gave a direct proof to show that as a Banach algebra with its left Arens product, the dual of a left introverted subspace of  $C_b(G)$  that contains  $C_0(G)$  determines  $G$ . In particular, it can be concluded that the dual of the left uniformly continuous functions on a locally compact group  $G$ ,  $LUC(G)^*$ , determines  $G$ . Their result also implies Johnson's aforementioned result. In [GLL90] Ghahramani, Lau and Losert showed that the second dual of the group algebra also determines the underlying group. Recently, Dales, Lau and Strauss [DLS12] proved that  $M(G)^{**}$ , determines the locally compact group  $G$ . In this article among other results we prove that when  $G$  is discrete  $C_0(G)^\perp$ , the annihilator of  $C_0(G)$  in  $LUC(G)^*$ , determines  $G$  as a locally compact group.

Let  $G$  be a Hausdorff locally compact group and  $C_b(G)$  denote the Banach space of bounded continuous functions on  $G$  with the uniform norm. An element  $f$  in  $C_b(G)$  is called left uniformly continuous if  $g \mapsto l_g f$ , where  $l_g f(x) = f(gx)$ , is a continuous map from  $G$  to  $C_b(G)$ . Let  $LUC(G)$  denote the Banach space of left uniformly continuous functions with the uniform norm. The dual space of  $LUC(G)$ ,  $LUC(G)^*$ , with the following Arens-type product forms a Banach algebra

$$\langle m.n, f \rangle = \langle m, nf \rangle \quad \text{and} \quad nf(g) = \langle n, l_g(f) \rangle$$

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where  $m, n \in LUC(G)^*$ ,  $f \in LUC(G)$  and  $g \in G$ . The measure algebra  $M(G)$  can be identified with a subalgebra of  $LUC(G)^*$  via

$$\langle \mu, f \rangle = \int_G f \, d\mu \quad (\mu \in M(G), f \in LUC(G)).$$

In fact we have the  $L^1$ - direct sum

$$LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp$$

where

$$C_0(G)^\perp := \{m \in LUC(G)^* : \langle m, f \rangle = 0, \forall f \in C_0(G)\}.$$

Moreover,  $C_0(G)^\perp$  is a weak-star closed ideal in  $LUC(G)^*$  (see [GLL90, Lemma 1.1]). One can easily check that for each  $n \in LUC(G)^*$  and  $\mu \in M(G)$  the mappings  $m \mapsto m \cdot n$  and  $n \mapsto \mu \cdot n$  are weak-star continuous. In fact,  $M(G)$  is the largest subset of elements  $m$  in  $LUC(G)^*$  for which the left multiplication mapping  $n \mapsto m \cdot n$ ,  $m \in LUC(G)^*$  is continuous (see [Lau86]).

The left uniformly continuous compactification of  $G$ ,  $G^{luc}$ , is the Gelfand spectrum of the unital commutative  $C^*$ - algebra  $LUC(G)$ , that is

$$G^{luc} := \{m \in LUC(G)^* \setminus \{0\} : \langle m, fg \rangle = \langle m, f \rangle \langle m, g \rangle \forall f, g \in LUC(G)\}.$$

It can be shown that  $G^{luc}$  is in fact a weak-star compact semigroup with the Arens multiplication and weak-star topology it inherits from  $LUC(G)^*$ . When  $G$  is a discrete group,  $LUC(G) = l^\infty(G) = C(\beta G)$  and so the  $LUC$ -compactification of  $G$  is the same as its Stone-Ćech compactification. We identify  $G$  with its image in  $G^{luc}$ . The corona of the  $LUC$ -compactification of  $G$ ,  $G^{luc} \setminus G$ , is denoted by  $G^*$  and is a closed ideal of the compact semigroup  $G^{luc}$ . An element  $z \in G^*$  is called right cancellable if for each  $m, n \in G^{luc}$ ,  $mz = nz$  implies that  $m = n$ . In [FP03, Thm. 1], it is proved that the  $LUC$ - compactification contains many right cancellable elements.

Let  $G$  and  $H$  be locally compact groups and  $\mathbb{T}$  be the circle group. Suppose that  $\alpha : G \rightarrow \mathbb{T}$  is a continuous character and  $\psi : G \rightarrow H$  is a continuous homomorphism. Then it is easy to see that

$$j_{\alpha, \psi}(f) := \alpha \cdot f \circ \psi$$

maps  $C_0(H)$  into  $C_0(G)$  and the dual mapping

$$j_{\alpha, \psi}^* : M(G) \rightarrow M(H)$$

is a homomorphism. When  $\psi$  is an isomorphism  $j_{\alpha, \psi}^* : M(G) \rightarrow M(H)$  is a weak-star continuous isometric isomorphism. It follows from [Joh64] that every isometric isomorphism  $T : M(G) \rightarrow M(H)$  is of the form  $T = j_{\alpha, \psi}^*$ , for some character  $\alpha : G \rightarrow \mathbb{T}$  and isomorphism  $\psi : G \rightarrow H$ , and therefore is weak-star continuous. Similarly, if  $\alpha : G \rightarrow \mathbb{T}$  is a continuous character and  $\psi : G \rightarrow H$  is a continuous homomorphism, then it is easy to see that

$$j_{\alpha, \psi}(f) := \alpha \cdot f \circ \psi$$

maps  $LUC(H)$  into  $LUC(G)$  and that the dual map

$$j_{\alpha,\psi}^* : LUC(G)^* \rightarrow LUC(H)^*$$

is a homomorphism. When  $\psi$  is a topological isomorphism  $j_{\alpha,\psi}^* : LUC(G)^* \rightarrow LUC(H)^*$  is a weak-star continuous isometric isomorphism. Moreover, every weak-star continuous isometric isomorphism  $T : LUC(G)^* \rightarrow LUC(H)^*$  also takes this canonical form, but it is not clear if every isometric isomorphism  $T : LUC(G)^* \rightarrow LUC(H)^*$  is weak-star continuous.

Suppose that  $\psi : G \rightarrow H$  is a continuous homomorphism. Letting  $\tilde{\psi}$  denote the restriction of  $j_{1,\psi}^*$  to  $G^{luc}$ ,  $\tilde{\psi} : G^{luc} \rightarrow H^{luc}$  is the unique continuous homomorphism extending  $\psi$ . We note that when  $\psi$  is a topological isomorphism,  $\tilde{\psi}$  is a topological isomorphism of  $G^{luc}$  onto  $H^{luc}$ .

F-spaces were studied in detail by L. Gillman and M. Henriksen in 1956 [GH56] as the class of spaces for which  $C(X)$  is a ring in which every finitely generated ideal is a principal ideal. Several conditions both topological and algebraic were proved equivalent for a space to be an F-space (see [GJ76]). We choose the following characterization as our definition for an F-space.

**Definition 1.1.** A completely regular space  $X$  is an F-space if for any continuous bounded function  $f$  on  $X$  there is a continuous bounded function  $k$  on  $X$  such that  $f = k|f|$ .

Many of the proofs in the case of discrete groups use the fact that for a discrete space the Stone-Ćech compactification and its corona are F-spaces (see for example [PPS00], [Fil], [Zel99], [MS96], [HS94] and [HS12]). This is especially useful due to the following lemma. A proof is given in [HS94, Lemma 1.1].

**Lemma 1.2.** *If  $X$  is a compact space then  $X$  is an F-space if and only if for  $\sigma$ -compact subsets  $A$  and  $B$  of  $X$ ,  $\bar{A} \cap B = \emptyset$  and  $A \cap \bar{B} = \emptyset$  implies that  $\bar{A} \cap \bar{B} = \emptyset$ .*

In the first section of this article we prove that for a locally compact non-discrete group neither the  $LUC$ -compactification nor its corona is an F-space. So in fact, for locally compact groups the corona of the  $LUC$ -compactification is an F-space if and only if the group is discrete. This shows that some interesting facts concerning Stone-Ćech compactifications of discrete groups cannot possibly be generalized to  $LUC$ -compactifications of general locally compact groups following the same line of proof. We also show that the corona of the  $LUC$ -compactification of a locally compact group does not contain  $P$ -points. A definition of  $P$ -points and a short discussion about their importance is given in the next section. In Section 3 we prove that a discrete group  $G$  is completely determined within the class of all locally compact groups by both  $C_0(G)^\perp$  and  $G^*$ . Some related results are also obtained.

## 2. WHEN IS $G^*$ AN F-SPACE?

Our main goal in this section is to show that if  $G$  is a locally compact non-discrete group then neither  $G^{luc}$  nor  $G^*$  are F-spaces. Therefore  $G^{luc}$  ( $G^*$ ) is an F-space if

and only if  $G$  is discrete. This result will be applied in Section 3.

In the proof of Theorem 2.2 we make use of absolutely convergent series with alternating partial sums. An example of such a series is given below.

**Example 2.1.** Let  $\sum_{n=1}^{\infty} b_n$  be a convergent series with positive terms and let

$$\begin{aligned} a_1 &:= b_1 \\ a_2 &:= -b_1 - b_2 \\ a_3 &:= b_2 + b_3 \\ a_4 &:= -b_3 - b_4 \\ &\dots \\ a_{2k} &:= -b_{2k-1} - b_{2k} \\ a_{2k+1} &:= b_{2k} + b_{2k+1} \\ &\dots \end{aligned}$$

Then the series  $\sum_{n=1}^{\infty} a_n$  is an example of an absolutely convergent series whose partial sums have alternating sign.

**Theorem 2.2.** *If  $X$  is a Hausdorff locally compact non-discrete topological space with a non-trivial convergent sequence, then  $X$  is not an  $F$ -space.*

*Proof.* Suppose that  $X$  contains a non-trivial sequence  $(x_n)$  convergent to a (non-isolated) point  $x_0$  in  $X$ . Without loss of generality assume that  $x_n \neq x_0$  for each  $n \in \mathbb{N}$ . Since  $x_0$  is not isolated and  $X$  is locally compact we can inductively construct a nested family  $\{K_n\}_{n \geq 2}$  of compact neighbourhoods around  $x_0$  such that  $x_1, x_2, \dots, x_n \notin K_{n+1}$  and  $x_{(n+1)}, x_{(n+2)}, \dots \in K_{n+1}$ , for each  $n$ . To see this note that since  $x_1 \neq x_0$  there is a precompact open set  $U_0$  such that  $\{x_2, x_3, \dots\} \cup \{x_0\} \in U_0$  and  $x_1 \notin U_0$ . Since  $X$  is a locally compact space for the compact set  $\{x_0\}$  and the open set  $U_0$  there is a compact set  $K_2$  such that  $\{x_0\} \subseteq K_2^\circ \subseteq K_2 \subseteq U_0$ , where  $K_2^\circ$  denotes the interior of the set  $K_2$ . Without loss of generality we can assume that  $(x_n)_{n \geq 2} \in K_2$ . Suppose that the compact set  $K_n$  is given such that  $x_1, \dots, x_{n-1} \notin K_n$ . Since  $x_1, \dots, x_n \neq x_0$  there is a precompact open set  $U_n$  such that  $\{x_{n+1}, x_{n+2}, \dots\} \cup \{x_0\} \subseteq U_n \subseteq K_n$  and  $x_1, \dots, x_n \notin U_n$ . Since  $X$  is locally compact there is a compact set  $K_{n+1}$  such that  $\{x_0\} \subseteq K_{n+1}^\circ \subseteq K_{n+1} \subseteq U_n$  and note that  $x_1, \dots, x_n \notin K_{n+1}$ . By discarding some elements of our sequence if necessary we can assume that  $x_{(n+1)}, x_{(n+2)}, \dots \in K_{n+1}$ .

Consider an absolutely convergent series  $\sum_{n=1}^{\infty} a_n$ , with alternating partial sums as in Example 2.1. Using Urysohn's lemma for locally compact spaces, for  $n \geq 2$  we define the compactly supported function  $f_n$  such that  $f_n(X \setminus K_n) = 0$  and  $f_n(K_{n+1} \cup \{x_n\}) = a_{n-1}$ , for  $n \geq 2$ . Let  $f := \sum_{n \geq 2} f_n$ . By the Weierstrass M-test, the series  $\sum_{n \geq 2} f_n$  is uniformly convergent to a continuous function. Note that  $f(X \setminus K_2) = 0$ . For any function  $k$  where  $f = k|f|$ , we observe that  $k$  is not continuous at  $x_0$ . To see this we note that  $f(x_n) = \sum_{m=1}^n a_m$ ,  $n \geq 2$ . Therefore,

$f(x_n)f(x_{n+1}) < 0$  as  $\sum_{m=1}^n a_m \sum_{m=1}^{n+1} a_m < 0$ . So the function  $k$  alternates on the convergent sequence  $(x_n)$  between  $+1$  and  $-1$  and it is not continuous at  $x_0$ .  $\square$

So to show that  $G^{luc} (G^*)$  is not an F-space, it is enough to show that it contains a non-trivial convergent sequence.

Kuzminov has shown that any compact group is dyadic, i.e., a continuous image of a Cantor cube. This implies that every infinite compact group contains a non-trivial convergent sequence. This result can be found for example in J. Van Mill's article in [Pea07] (see page 190).

**Theorem 2.3.** *If  $G$  is a locally compact non-discrete group then  $G$  has a non-trivial convergent sequence. In particular,  $G$  is not an F-space.*

*Proof.* It is easy to see that if  $G$  is a metrizable non-discrete locally compact group then  $G$  has a non-trivial convergent sequence. If  $G$  is a locally compact non-discrete group then  $G$  has a subgroup  $H$  that is sigma-compact, clopen and non-discrete. If  $H$  is metrizable then any point in  $H$  is a limit point of a non-trivial convergent sequence in  $H$ . This sequence is also convergent in  $G$ . So suppose that  $H$  is not metrizable. By the Kakutani-Kodaira theorem [HR79, Thm. 8.5] there is a compact normal subgroup  $N$  of  $H$  such that  $H/N$  is metrizable, and so by [HR79, 5.38 part (e)],  $N$  cannot be metrizable since otherwise  $H$  would be metrizable. Because  $N$  is not metrizable it cannot be finite. As noted above, since  $N$  is compact and non-discrete we can find a non-trivial convergent sequence in  $N$ . Any such non-trivial convergent sequence is also convergent in the open subgroup  $H$  and therefore in  $G$ .  $\square$

As stated in the introduction, were  $G^*$  an F-space for some non-discrete groups, we would be able to prove stronger versions of our main results in the next section namely, Theorem 3.7 and Corollary 3.8. However, as the next result shows,  $G^*$  is an F-space if and only if  $G$  is discrete.

**Theorem 2.4.** *Suppose that  $G$  is a locally compact non-discrete group. Then neither  $G^{luc}$  nor  $G^*$  is an F-space.*

*Proof.* Suppose that  $G$  is a locally compact non-discrete group. Then  $G$  contains a non-trivial sequence, say  $(x_n)$ , convergent to an element  $x$ . Let  $z$  be a right cancellable element in  $G^*$ . Then the non-trivial sequence  $(x_n z)$  converges to  $xz$ . So in both cases, neither  $G^{luc}$  nor  $G^*$  is an F-space, by Theorem 2.2.  $\square$

**Definition 2.5.** A point in a topological space is called a  $P$ -point if every  $G_\delta$ -set containing the point is a neighbourhood of the point.

Gillman and Henriksen [GJ76] were the first to study  $P$ -points. If  $G$  is a discrete group then under the continuum hypothesis the set of  $P$ -points in  $G^*$  forms a dense subset in  $G^*$ . It is a fact that the existence of  $P$ -points cannot be proved in  $ZFC$ . We refer the reader to [HS12] and the remark in [Fil96, page 385] for an explanation of these statements. From Corollary 2.6 (below) we see that under the continuum

hypothesis, if  $G$  is a locally compact group, then  $G^{luc} \setminus G$  has a  $P$ -point if and only if  $G$  is discrete.  $P$ -points were used in [Fil, Thm. 3] to show that for a discrete group  $G$  there are left cancellable elements. In fact these particular  $P$ -points in [Fil, Thm. 3] are also right cancellable. It is not known if for the general case of discrete groups there can be left cancellable elements that are not right cancellable (see [HS12, Thm. 8.40]).

**Corollary 2.6.** *Suppose that  $G$  is a locally compact non-discrete group. Then  $G^*$  does not contain any  $P$ -point.*

*Proof.* First observe that for each point  $p$  in  $G^*$  there is a non-trivial sequence  $(x_n)$  converging to  $p$ . To see this, note that from the proof of Lemma 2.3 there is a non-trivial sequence  $(z_n)$ , convergent to a point  $y$  in the group. Therefore, the non-trivial sequence  $(y_n)$  where  $y_n := y^{-1}z_n$  is convergent to  $e$ . Now  $x_n := y_np$  is a non-trivial sequence convergent to  $p$  in  $G^*$ . Let  $\{U_n\}$  be a family of open neighbourhoods of  $p$  such that for each  $n$ ,  $x_1, x_2, \dots, x_n \notin U_n$ . Then  $p \in \cap_m U_m$ , but for each  $n$ ,  $x_n \notin \cap_m U_m$  so  $\cap_m U_m$  cannot be open. Hence  $p$  is not a  $P$ -point.  $\square$

### 3. ISOMORPHISM ON $C_0(G)^\perp$

In [GLL90] Ghahramani, Lau and Losert showed that given locally compact groups  $G$  and  $H$ , every isometric isomorphism  $T : LUC(G)^* \rightarrow LUC(H)^*$  maps  $M(G)$  onto  $M(H)$ . It can be shown directly that  $G^{luc}$  determines  $G$ , within the class of locally compact groups. To see this we note that  $G^*$  is an ideal in  $G^{luc}$  and therefore the only invertible elements of  $G^{luc}$  are elements of  $G$ . An interesting question is whether  $C_0(G)^\perp (G^*)$  also determines  $G$ . In this section we employ Theorem 2.4 to show that this is the case when  $G$  is discrete (Theorem 3.7 and Corollary 3.8).

First we show that if  $G$  and  $H$  are non-compact locally compact groups and if  $T : LUC(G)^* \rightarrow LUC(H)^*$  is an isometric isomorphism, then  $C_0(G)^\perp$  is also mapped onto  $C_0(H)^\perp$ . We recall the following lemma from [Str65].

**Lemma 3.1.** *Let  $X$  be a locally compact space, and let  $\mu$  and  $\nu \in M(X)$ . Then  $\mu$  and  $\nu$  are mutually singular if and only if  $\|\mu + \nu\| = \|\mu - \nu\| = \|\mu\| + \|\nu\|$ .*

Lemma 3.1 implies that isometries preserve mutual singularity.

Since  $LUC(G) = C(G^{luc})$ , the Banach space of continuous functions on  $G^{luc}$ , we have that  $LUC(G)^* = M(G^{luc})$ , the Banach space of all regular Borel measures on  $G^{luc}$ . It can be seen that  $C_0(G)^\perp$  is isometrically isomorphic to  $M(G^*)$ , the Banach space of regular Borel measures on  $G^*$ . Therefore,  $M(G^{luc}) = LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp = M(G) \oplus_1 M(G^*)$ .

**Theorem 3.2.** *Suppose that  $G$  and  $H$  are non-compact locally compact groups. If  $T : LUC(G)^* \rightarrow LUC(H)^*$  is an isometric isomorphism. Then  $T$  maps  $C_0(G)^\perp$  onto  $C_0(H)^\perp$ .*

*Proof.* Let  $m \in C_0(G)^\perp$ , we show that  $T(m) \in C_0(H)^\perp$ . Suppose that  $T(m) = \nu + r$ , where  $\nu \in M(H)$  and  $r \in C_0(H)^\perp$ , so that  $\nu$  and  $r$  are mutually singular measures in  $M(H^{luc})$ . Since an isometry preserves mutual singularity, if we consider  $m = T^{-1}(\nu) + T^{-1}(r)$ , then we have that  $T^{-1}(\nu)$  and  $T^{-1}(r)$  are singular and also we have that  $T^{-1}(\nu) \in M(G)$ , by [GLL90, Thm. 1.6]. Suppose that  $T^{-1}(r) = \nu' + r'$  where  $\nu' \in M(G)$  and  $r' \in C_0(G)^\perp$ . Therefore  $m = T^{-1}(\nu) + \nu' + r'$ . Now, since  $T^{-1}(\nu), \nu' \in M(G)$  we must have  $T^{-1}(\nu) + \nu' = 0$  because  $M(G) \cap C_0(G)^\perp = 0$ . Since  $\nu'$  is absolutely continuous with respect to  $T^{-1}(r)$ , it is mutually singular with  $T^{-1}(\nu)$ , and so we have that  $T^{-1}(\nu) = 0$ . Hence  $T(m) = \nu + r = r \in C_0(H)^\perp$ .  $\square$

**Corollary 3.3.** *Suppose that  $G$  and  $H$  are locally compact groups. Then  $T : LUC(G)^* \rightarrow LUC(H)^*$  is an isometric isomorphism if and only if there are isometric isomorphisms  $T_1 : M(G) \rightarrow M(H)$  and  $T_2 : C_0(G)^\perp \rightarrow C_0(H)^\perp$  such that  $T = T_1 + T_2$ ,  $T_2(\mu m) = T_1(\mu)T_2(m)$  and  $T_2(m\mu) = T_2(m)T_1(\mu)$  for all  $m \in C_0(G)^\perp$  and  $\mu \in M(G)$ .*

Note that Corollary 3.3 shows that the value of  $T_1$  is connected to that of  $T_2$ . It is not clear if every isometric isomorphism  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  can be extended to one on  $LUC(G)^*$ . When either  $G$  or  $H$  is abelian and discrete and  $T$  is weak-star continuous, we will show that such an extension always exists (Theorem 3.11). Proposition 3.5 shows that such an extension is always unique. We need the following lemma for the proof of Proposition 3.5.

**Lemma 3.4.** *Suppose that  $G$  is a locally compact group and let  $z \in G^*$  be a right cancellable in  $G^{luc}$ . The following statements hold:*

- (i)  *$z$  is also right cancellable in  $LUC(G)^*$ .*
- (ii) *If  $(n_i)$  is a bounded net in  $LUC(G)^*$  and  $n_i z \xrightarrow{w^*} nz$  in  $LUC(G)^*$ , then  $n_i \xrightarrow{w^*} n$  in  $LUC(G)^*$ .*

*Proof.* (i) Let  $z \in G^*$  be right cancellable in  $G^{luc}$ . Then we show that  $z$  is also right cancellable in  $LUC(G)^*$ . To see this we note that since  $z$  is right cancellable, the unital  $*$ -algebra

$$\{zf, f \in LUC(G)\} \quad \text{where} \quad zf(x) = \langle z, l_x f \rangle$$

separates the points in  $LUC(G)$  and thus, by the Stone-Weierstrass theorem, is dense in  $LUC(G) = C(G^{luc})$ . So if for some  $m, n \in LUC(G)^*$  we have that  $mz = nz$ , then for all  $f \in LUC(G)$  we must have  $\langle m, zf \rangle = \langle n, zf \rangle$  and therefore  $m = n$ .

(ii) Suppose that  $(n_i)$  is a net in  $LUC(G)^*$  such that  $(n_i)$  is bounded in norm by  $M > 0$  and  $n_i z \xrightarrow{w^*} nz$  in  $LUC(G)^*$ . We show that  $n_i \xrightarrow{w^*} n$  in  $LUC(G)^*$ . Suppose that  $\varepsilon > 0$  and  $f \in LUC(G)$  are given. As noted in the proof of part (i), the algebra  $\{zg, g \in LUC(G)\}$  is norm-dense in  $LUC(G)$  so there is  $g \in LUC(G)$  such that

$$\|f - zg\| \leq \frac{\varepsilon}{M}.$$

Also  $n_i z \xrightarrow{w^*} nz$ , so there is  $i_0$  such that for all  $i \geq i_0$  we have

$$|n_i z(g) - nz(g)| \leq \varepsilon.$$

Therefore for all  $i \geq i_o$ ,

$$\begin{aligned} |n_i(f) - n(f)| &\leq |n_i(f) - n_i(zg)| + |n_i(zg) - n(zg)| \\ &\quad + |n(zg) - n(f)| \\ &\leq \varepsilon + \varepsilon + \varepsilon. \end{aligned}$$

□

**Proposition 3.5.** *Suppose that  $G$  and  $H$  are locally compact groups and  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is an (algebraic) isomorphism. If there is an isomorphism  $\tilde{T} : LUC(G)^* \rightarrow LUC(H)^*$  such that  $\tilde{T}$  is an extension of  $T$ , then  $\tilde{T}$  is unique.*

*Proof.* Suppose that  $T_1$  and  $T_2$  are two such extensions. Let  $z \in G^*$  be right cancellable in  $G^{luc}$ . Then, by Lemma 3.4,  $z$  is also right cancellable in  $LUC(G)^*$  and hence,  $T(z) = T_1(z) = T_2(z)$  is also right cancellable in  $LUC(H)^*$ . For each  $m$  in  $LUC(G)^*$ , then we have that

$$T_1(m)T(z) = T(mz) = T_2(m)T(z).$$

Since  $T(z)$  is right cancellable we have that  $T_1(m) = T_2(m)$ , for all  $m$  in  $LUC(G)^*$ . □

The following proposition shows that under certain conditions the extension of an algebraic homomorphism  $\phi : G^* \rightarrow H^*$  to an algebraic homomorphism  $\varphi : G^{luc} \rightarrow H^{luc}$  is also unique.

**Proposition 3.6.** *Suppose that  $G$  and  $H$  are locally compact groups and  $\phi : G^* \rightarrow H^*$  is an algebraic homomorphism. Suppose that  $\varphi_L, \varphi_{L'} : G^{luc} \rightarrow H^{luc}$  are homomorphic extensions of  $\phi$  such that either*

- (i)  $\varphi_L(G), \varphi_{L'}(G) \subseteq H$ , or
- (ii) the interior of  $\phi(G^*)$  is non-empty.

*Then,  $\varphi_L = \varphi_{L'}$ .*

*Proof.* Let  $x \in G$ . Then, for all  $p \in G^*$ ,

$$\varphi_L(x)\phi(p) = \varphi_L(xp) = \phi(xp) = \varphi_{L'}(xp) = \varphi_{L'}(x)\phi(p).$$

In the case of (i),  $\varphi_L(x) = \varphi_{L'}(x)$ , by Veech's theorem [Pym99]. By [FP03, Thm. 1], the set of right cancellable elements is dense in  $G^*$ , so in the second case we can choose  $p \in G^*$  such that  $\phi(p)$  is right cancellable. □

In fact, if  $\varphi : G^{luc} \rightarrow H^{luc}$  is a surjective homomorphism, then  $\varphi$  satisfies condition (i) in Proposition 3.6. To see this, we observe that since  $\varphi$  is onto, there is a  $q \in G^{luc}$ , such that  $e_H = \varphi(q)$ . We have that

$$e_H = \varphi(e_G q) = \varphi(e_G)\varphi(q) = \varphi(e_G)e_H = \varphi(e_G).$$

Thus, for each  $x$  in  $G$ , we have that  $e_H = \varphi(x)\varphi(x^{-1})$  and thus  $\varphi(x)$  must belong to  $H$ .

As noted in the introduction, when  $\psi : G \rightarrow H$  is a continuous homomorphism



and  $\alpha : G \rightarrow \mathbb{T}$  is the constant character 1,  $j_{1,\psi}^* : LUC(G)^* \rightarrow LUC(H)^*$  is a weak-star continuous homomorphism and  $\tilde{\psi} := j_{1,\psi}^*|_{G^{luc}}$  is a continuous homomorphism of  $G^{luc}$  into  $H^{luc}$ . We say that the continuous homomorphism  $\phi : G^* \rightarrow H^*$  is induced by a continuous homomorphism  $\psi : G \rightarrow H$ , if  $\tilde{\psi}|_{G^*} = \phi$ .

Note that if  $\phi : G^* \rightarrow H^*$  is a continuous homomorphism, it is not necessarily induced by a homomorphism  $\psi : G \rightarrow H$ . A simple example is  $\phi : G^* \rightarrow H^*$  where  $\phi(p) = \iota$  and  $\iota$  is an idempotent in  $H^*$  (see [HS12, Thm. 2.5] for a proof of the existence of idempotents in  $G^*$ ). To see this suppose that there is a homomorphism  $\psi : G \rightarrow H$  such that  $\tilde{\psi}|_{G^*} = \phi$ . Then since

$$\iota = \tilde{\psi}(xp) = \psi(x)\phi(p) = \psi(x)\iota$$

for (any)  $p \in G^*$  and  $x \in G$ , we have that  $\psi$  is the trivial homomorphism  $x \mapsto e_G$ , by Veech's theorem [Pym99]. By uniqueness,  $\tilde{\psi}(p) = e_G$ , for all  $p \in G^{luc}$ , which is not possible as  $\tilde{\psi}|_{G^*} = \phi$ .

We apply Theorem 2.4 and [PPS00, Thm. 6.2] to prove the next result.

**Theorem 3.7.** *Let  $G$  and  $H$  be locally compact groups. Suppose that  $\phi : G^* \rightarrow H^*$  is a continuous isomorphism and that either  $G$  or  $H$  is discrete. Then there is a unique topological isomorphism  $\psi : G \rightarrow H$  such that  $\phi = \tilde{\psi}|_{G^*}$ . In particular,  $G$  and  $H$  are isomorphic, as topological groups.*

*Proof.* Since  $\phi : G^* \rightarrow H^*$  is a continuous isomorphism, by Theorem 2.4 we have that both  $G$  and  $H$  are discrete groups. The existence of such a continuous surjection  $\psi : G \rightarrow H$  now follows from [PPS00, Thm. 6.2]. We shall also show that  $\psi$  is injective. Suppose that for some  $x, y \in G$ , we have that  $\psi(x) = \psi(y)$ . Then for each  $q \in G^*$ , since  $H^*$  is an ideal in  $H^{luc}$ , we have that

$$\phi(xq) = \tilde{\psi}(xq) = \psi(x)\phi(q) = \psi(y)\phi(q) = \tilde{\psi}(yq) = \phi(yq).$$

Since  $\phi$  is injective,  $xq = yq$  and so, by Veech's theorem [Pym99],  $x = y$ . Thus  $\psi$  is an isomorphism of topological groups and therefore  $G$  and  $H$  must be isomorphic.  $\square$

Is [PPS00, Thm. 6.2] true when  $G$  and  $H$  are not assumed to be discrete? Although some details such as the existence of prime elements in the proof of [PPS00, Thm. 6.2] remain valid for non-discrete locally compact groups, the proof heavily depends on the Lemma 1.2 above. By Theorem 2.4, Lemma 1.2 cannot be employed for non-discrete locally compact groups, so the same proof will not work.

**Corollary 3.8.** *Let  $G$  and  $H$  be locally compact groups. Suppose that  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is a weak-star continuous isometric isomorphism and either  $G$  or  $H$  is discrete. Then the topological groups  $G$  and  $H$  are isomorphic.*

*Proof.* First we note that since  $C_0(G)^\perp = M(G^*)$ , by [Con90, Thm. V.8.4] the set of extreme points of the unit ball of  $C_0(G)^\perp$  is

$$\{\alpha\delta_p; \alpha \in \mathbb{T}, p \in G^*\}.$$

In particular the points in  $G^*$  are among the extreme points of the unit ball of  $C_0(G)^\perp$ . Because  $T$  is an isometry, it maps the extreme points of the unit ball of

$C_0(G)^\perp$  onto the extreme points of the unit ball of  $C_0(H)^\perp$ . Therefore there exist maps  $\phi : G^* \rightarrow H^*$  and  $\alpha : G^* \rightarrow \mathbb{T}$  such that  $T(\delta_p) = \alpha(p)\delta_{\phi(p)}$ , for all  $p \in G^*$ . We first show that  $\alpha$  is continuous. Suppose that  $(p_\gamma)$  is a net in  $G^*$  that is convergent to  $p \in G^*$ . Since  $T$  is weak-star continuous we have that

$$\alpha(p_\gamma)\delta_{\phi(p_\gamma)} = T(\delta_{p_\gamma}) \xrightarrow{w^*} T(\delta_p) = \alpha(p)\delta_{\phi(p)}.$$

Evaluating this equation at  $1_{G^*}$  implies that  $\alpha(p_\gamma) \rightarrow \alpha(p)$ . Hence  $\alpha$  is continuous. Similarly, since  $T$  is an isomorphism we can show that  $\alpha$  is also multiplicative and thus  $\alpha$  is in fact a continuous character. Therefore  $\phi = \bar{\alpha}T|_{G^*} : G^* \rightarrow H^*$  is a continuous isomorphism. Since  $G^*$  is compact and  $H^*$  is Hausdorff, the continuous isomorphism  $\phi : G^* \rightarrow H^*$  is also a homeomorphism. Now the result follows from Theorem 3.7.  $\square$

When  $G$  and  $H$  are both non-discrete locally compact groups, it remains open whether  $C_0(G)^\perp$  determines  $G$ .

**Proposition 3.9.** *Let  $G$  be a locally compact group,  $K$  a compact group, and suppose that  $\alpha : G^* \rightarrow K$  is a continuous homomorphism. Let  $\iota$  be any idempotent in  $G^*$  and define  $\alpha_\iota : G^{luc} \rightarrow K$  by  $\alpha_\iota(x) = \alpha(x\iota)$ . Then the following statements hold:*

- (i)  $\alpha_\iota$  is a continuous extension of  $\alpha$  to  $G^{luc}$ .
- (ii) If  $\beta$  is any homomorphic extension of  $\alpha$  to  $G^{luc}$ , then  $\beta = \alpha_\iota$ . (Thus any such homomorphic extension -if it exists- is unique and automatically continuous.)
- (iii) If  $\iota$  commutes with elements of  $G$ , then  $\alpha_\iota$  is a homomorphism on  $G^{luc}$ .

*Proof.* (i) We note that

$$\alpha(\iota) = \alpha(\iota^2) = \alpha(\iota)^2,$$

so  $\alpha(\iota) = e_K$ , the identity in  $K$ . As  $x \mapsto x\iota$  is a continuous mapping of  $G^{luc}$  into  $G^*$ , and  $\alpha$  is continuous on  $G^*$ ,  $\alpha_\iota$  is continuous on  $G^{luc}$ .

(ii) If  $\beta$  is any such homomorphic extension of  $\alpha$ , then for any  $x \in G^{luc}$ ,

$$\beta(x) = \beta(x)e_K = \beta(x)\beta(\iota) = \beta(x\iota) = \alpha(x\iota) = \alpha_\iota(x).$$

(iii) For any  $x, y \in G$ ,

$$\alpha_\iota(xy) = \alpha(xy\iota) = \alpha(xy\iota^2) = \alpha(x\iota y\iota) = \alpha(x\iota)\alpha(y\iota) = \alpha_\iota(x)\alpha_\iota(y).$$

Thus,  $\alpha_G := \alpha|_G : G \rightarrow K$  is a continuous homomorphism. (Note that it is not yet clear that  $\alpha_\iota$  is a homomorphism on  $G^{luc}$ , even though its restrictions to  $G$  and  $G^*$  are homomorphisms.) As observed in the introduction,  $\alpha_G$  extends to a continuous homomorphism  $\widetilde{\alpha}_G : G^{luc} \rightarrow K$  (note that since  $K$  is compact,  $K^{luc} = K$ ). As both  $\widetilde{\alpha}_G$  and  $\alpha_\iota$  are continuous extensions of  $\alpha_G = \alpha|_G$  to  $G^{luc}$ , we must have  $\alpha_\iota = \widetilde{\alpha}_G$ . Hence,  $\alpha_\iota$  is a homomorphism.  $\square$

**Corollary 3.10.** *If  $G$  is an abelian locally compact group and  $K$  is a compact group, then every continuous homomorphism  $\alpha : G^* \rightarrow K$  has a unique continuous extension to  $G^{luc}$ .*

*Proof.* Let  $\iota$  be any idempotent in  $G^*$  and  $(x_\gamma)$  be a net in  $G$  convergent to  $\iota$ . We have that for each  $x \in G$

$$x\iota = x(\lim_\gamma x_\gamma) = \lim_\gamma (xx_\gamma) = \lim_\gamma (x_\gamma x) = (\lim_\gamma x_\gamma)x = \iota x.$$

The result now follows from Proposition 3.9.  $\square$

We say a linear operator  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is positive if for each positive linear functional  $m \in M(G^*) = C(G^*)^*$  we have that  $T(m)$  is a positive linear functional, here as usual  $m$  positive means  $m(f) \geq 0$ , whenever  $f \geq 0$ .

**Theorem 3.11.** *Let  $G$  and  $H$  be locally compact groups with either  $G$  or  $H$  discrete and suppose that  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is a weak-star continuous isometric isomorphism. If either  $G$  is abelian, or  $T$  is a positive operator, then there exists a unique weak-star continuous isometric isomorphism  $\tilde{T} : LUC(G)^* \rightarrow LUC(H)^*$  such that  $\tilde{T}|_{C_0(G)^\perp} = T$ .*

*Proof.* The uniqueness will follow from Proposition 3.5. The proof of Corollary 3.8 shows that there exists a continuous character  $\alpha : G^* \rightarrow \mathbb{T}$  and a topological isomorphism  $\phi : G^* \rightarrow H^*$  such that

$$T(\delta_x) = \alpha(x)\delta_{\phi(x)} \quad (x \in G^*).$$

By Theorem 3.7, there exists a topological isomorphism  $\psi : G \rightarrow H$  such that  $\tilde{\psi}|_{G^*} = \phi$ . If  $G$  is abelian, by Corollary 3.10, there exists a unique continuous character  $\alpha_G : G \rightarrow \mathbb{T}$  such that  $\tilde{\alpha}_G|_{G^*} = \alpha$ ; if  $T$  is positive,  $\alpha \equiv 1_{G^*}$  and  $\alpha = \tilde{1}_G|_{G^*}$ . As noted in the introduction,  $\tilde{T} = j_{\alpha_G, \psi}^*$  is a weak-star continuous isometric isomorphism of  $LUC(G)^*$  onto  $LUC(H)^*$ . For  $x \in G^*$ , weak-star continuity and density considerations give

$$\tilde{T}(\delta_x) = \alpha(x)\delta_{\phi(x)} = T(\delta_x)$$

and the proof is complete.  $\square$

Suppose that  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is an isometric isomorphism (not necessarily weak-star continuous). Then for each  $x \in H$ , the mapping

$$L_x : C_0(G)^\perp \rightarrow C_0(G)^\perp, \quad \text{where} \quad L_x(m) = T^{-1}(\delta_x T(m))$$

is an invertible isometric left multiplier on  $C_0(G)^\perp$  (i.e.  $L_x(mn) = L_x(m)n$ ,  $m, n \in C_0(G)^\perp$ ) with inverse  $L_{x^{-1}}$ . Similarly, for each  $x \in H$ , the mapping

$$R_x : C_0(G)^\perp \rightarrow C_0(G)^\perp, \quad \text{where} \quad R_x(m) = T^{-1}(T(m)\delta_x)$$

is an invertible isometric right multiplier on  $C_0(G)^\perp$ . We call  $L_x : C_0(G)^\perp \rightarrow C_0(G)^\perp$  a left-point multiplier and  $R_x : C_0(G)^\perp \rightarrow C_0(G)^\perp$  a right-point multiplier associated with  $T$ . The reader is referred to [Pal94, Sections 1.2.1-1.2.7] for definitions and basic theorems regarding the left/right multipliers.

Suppose that  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is a weak-star continuous isometric isomorphism. Moreover, suppose that  $T = \tilde{T}|_{C_0(G)^\perp}$ , where  $\tilde{T} = j_{\alpha,\psi}^* : LUC(G)^* \rightarrow LUC(H)^*$  for some character  $\alpha$  on  $G$  and some topological isomorphism  $\psi : G \rightarrow H$ . (By Theorem 3.11, this is the case when  $G$  is discrete and either  $G$  is abelian or  $T$  is positive.) Then given  $x \in H$ , for each  $m \in C_0(G)^\perp$

$$\begin{aligned} L_x(m) = T^{-1}(\delta_x T(m)) &= \tilde{T}^{-1}(\delta_x \tilde{T}(m)) \\ &= \tilde{T}^{-1}(\delta_x) \tilde{T}^{-1}(\tilde{T}(m)) \\ &= \bar{\alpha}(\psi^{-1}(x)) \delta_{\psi^{-1}(x)} m. \end{aligned}$$

We shall say that a multiplier  $L : C_0(G)^\perp \rightarrow C_0(G)^\perp$  is *given by a point-mass* if there exist  $y \in G$  and  $\gamma \in \mathbb{T}$  such that  $L(m) = \gamma \delta_y m$ , for all  $m \in C_0(G)^\perp$ . The above argument shows that left-point multipliers associated with “canonical form” isomorphisms  $T = j_{\alpha,\psi}^*$  (and their inverses) are given by point-masses. We now prove the converse of this statement.

**Theorem 3.12.** *Let  $G$  and  $H$  be locally compact groups and  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  be a weak-star continuous isometric isomorphism. Suppose that the left-point multipliers associated with  $T$  and  $T^{-1}$  are given by point-masses. Then  $T$  takes the canonical form  $T = j_{\beta,\gamma}^*$  for a character  $\beta$  on  $G$  and a topological isomorphism  $\gamma : G \rightarrow H$ . In particular,  $T$  extends to a topological isomorphism of  $LUC(G)^*$  onto  $LUC(H)^*$  and  $G$  and  $H$  are topologically isomorphic.*

*Proof.* For each  $x \in H$ , let  $\psi(x) \in G$  and  $\alpha(x) \in \mathbb{T}$  be such that  $L_x(m) = \alpha(x) \delta_{\psi(x)} m$ , for all  $m \in C_0(G)^\perp$ . Suppose that  $z \in G^*$  is right cancellable in  $LUC(G)^*$ . First we show that both  $\alpha$  and  $\psi$  are multiplicative. To see this, let  $x, y \in H$ . Then, we have that

$$\begin{aligned} \alpha(xy) \delta_{\psi(xy)} \delta_z = L_{xy}(\delta_z) &= T^{-1}(\delta_{xy} T(\delta_z)) \\ &= T^{-1}[\delta_x T(T^{-1}(\delta_y T(\delta_z)))] \\ &= \alpha(x) \delta_{\psi(x)} T^{-1}(\delta_y T(\delta_z)) \\ &= \alpha(x) \alpha(y) \delta_{\psi(x)} \delta_{\psi(y)} \delta_z = \alpha(x) \alpha(y) \delta_{\psi(x)\psi(y)} \delta_z, \end{aligned}$$

so  $\alpha(xy) = \alpha(x)\alpha(y)$  and  $\psi(xy) = \psi(x)\psi(y)$ . Now we show that  $\psi : H \rightarrow G$  is continuous. Suppose that  $x_\gamma \rightarrow x$  in  $H$ . Since  $T$  (and therefore  $T^{-1}$ ) is weak-star continuous, we have that  $L_{x_\gamma}(\delta_z) \rightarrow L_x(\delta_z)$  and so  $\alpha(x_\gamma) \delta_{\psi(x_\gamma)} \delta_z \xrightarrow{w^*} \alpha(x) \delta_{\psi(x)} \delta_z$  in  $C_0(G)^\perp$ . Evaluating at the constant function 1 in  $LUC(G)$ , we see that  $\alpha(x_\gamma) \rightarrow \alpha(x)$ , and therefore  $\delta_{\psi(x_\gamma)} \delta_z \xrightarrow{w^*} \delta_{\psi(x)} \delta_z$  in  $C_0(G)^\perp$ . It follows from Lemma 3.4 that  $\delta_{\psi(x_\gamma)} \xrightarrow{w^*} \delta_{\psi(x)}$ . Thus,  $\psi$  is continuous. To see that  $\psi$  is a bijection, we note that given  $y \in G$ ,  $\gamma(y) \in H$  and  $\beta(y) \in \mathbb{T}$  are such that

$$\beta(y) \delta_{\gamma(y)} m = T(\delta_y T^{-1}(m)) \quad (m \in C_0(H)^\perp),$$

then the above argument shows that  $\gamma$  and  $\beta$  are also continuous maps. Letting  $x \in G$  and  $m = T^{-1}(\delta_z) \in C_0(G)^\perp$ , we obtain

$$\begin{aligned} \beta(\psi(x))\delta_{\gamma(\psi(x))z} &= \beta(\psi(x))\delta_{\gamma(\psi(x))}\delta_z = T(\delta_{\psi(x)}T^{-1}(\delta_z)) \\ &= T\left(\overline{\alpha(x)}\alpha(x)\delta_{\psi(x)}m\right) \\ &= \overline{\alpha(x)}T(L_x(m)) \\ &= \overline{\alpha(x)}T(T^{-1}(\delta_x T(m))) \\ &= \overline{\alpha(x)}\delta_x\delta_z = \overline{\alpha(x)}\delta_{xz}. \end{aligned}$$

Hence  $\gamma = \psi^{-1}$  (and  $\beta \circ \psi = \bar{\alpha}$ ; equivalently  $\beta = \bar{\alpha} \circ \psi^{-1}$ ). Now we show that  $T$  has the desired canonical form. To see this, suppose that  $p \in H^{luc}$ . Let  $(x_i)$  be a net in  $H$  converging to  $p$ . We have that

$$\begin{aligned} \tilde{\alpha}(p)\delta_{\tilde{\psi}(p)}m &= \lim_i \alpha(x_i)\delta_{\psi(x_i)}m \\ &= \lim_i T^{-1}(\delta_{x_i}T(m)) \\ &= T^{-1}(\delta_p T(m)) = T^{-1}(\delta_p)m, \end{aligned}$$

for all  $m \in C_0(G)^\perp$ . Taking  $m = z \in G^*$ , a right cancellable element in  $G^{luc}$ , we have that  $T^{-1}(\delta_p) = \tilde{\alpha}(p)\delta_{\tilde{\psi}(p)}$ . So  $T^{-1}|_{G^*} = j_{\alpha,\psi}^*|_{G^*}$  and, since  $T^{-1}$  is weak-star continuous, we have that  $T^{-1} = j_{\alpha,\psi}^*|_{C_0(G)^\perp}$ . It follows that  $T$  has the canonical form  $T = j_{\bar{\alpha} \circ \psi^{-1}, \psi^{-1}}^*|_{C_0(G)^\perp} = j_{\beta,\gamma}^*|_{C_0(G)^\perp}$ , and therefore extends to  $LUC(G)^*$ .  $\square$

It can be shown that if  $G$  and  $H$  are locally compact groups then any isometric (algebra) isomorphism  $T : M(G) \rightarrow M(H)$  is weak-star continuous. It is thus natural to ask if every isometric isomorphism  $T : LUC(G)^* \rightarrow LUC(H)^*$ , equivalently  $T : M(G^{luc}) \rightarrow M(H^{luc})$ , is also weak-star continuous. In fact, this question was the motivation for this paper. We have shown that any such isometric isomorphism maps  $C_0(G)^\perp$  onto  $C_0(H)^\perp$ , so a related question is if every isometric isomorphism  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is weak-star continuous. Similar questions can be asked about algebraic isomorphisms  $\phi : G^{luc} \rightarrow H^{luc}$  and  $\varphi : G^* \rightarrow H^*$ .

We conclude with a discussion of the multipliers on  $C_0(G)^\perp$ , which may be useful with regards to the problems described above and which we think is of independent interest. Suppose that  $G$  is a locally compact group. Since  $LUC(G)^*$  is unital, a simple observation shows that every left multiplier  $L$  is of the form  $L(n) = m_0 n$ :

$$L(n) = L(\delta_e n) = L(\delta_e)n.$$

So, since  $LUC(G)^*$  is unital, it is easy to characterize its left multipliers. The left multiplier  $L$  is weak-star continuous if and only if  $m_0 \in M(G)$  (see [Lau86, Corollary 3]). Also,  $L : LUC(G)^* \rightarrow LUC(G)^*$  is an invertible isometric left multiplier if and only if  $m_0 = \alpha\delta_x$ ,  $\alpha \in \mathbb{T}$ ,  $x \in G$  (see [GLL90, Corollary 1.2]). So it is easy to characterize both the weak-star continuous and invertible isometric left multipliers on  $LUC(G)^*$ . Therefore, if we could show that every left-point multiplier on  $C_0(G)^\perp$  associated with a weak-star continuous isometric isomorphism  $T : C_0(G)^\perp \rightarrow C_0(G)^\perp$

extends to an isometric invertible left multiplier on  $LUC(G)^*$ , then every left-point multiplier on  $C_0(G)^\perp$  must be given by a point-mass; by Theorem 3.12 we could then conclude that  $C_0(G)^\perp$  determines  $G$ . Similarly, every right multiplier  $R$  on  $LUC(G)^*$  is of the form  $R(n) = nm_0$ , for some  $m_0 \in LUC(G)^*$ , and therefore, every right multiplier on  $LUC(G)^*$  is weak-star continuous. As shown in Proposition 3.13 below, the right-point multipliers  $R_x$  associated with  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  (as defined above) are also always weak-star continuous (without assuming that  $T$  is weak-star continuous).

When  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is an isomorphism and  $x \in H$ , it is readily verified that the pair  $(L_x, R_x)$  is a double centralizer of  $C_0(G)^\perp$  (i.e.  $mL_x(n) = R_x(m)n$  for all  $m, n \in C_0(G)^\perp$ ). We note that if  $(L, R)$  is a double centralizer of  $C_0(G)^\perp$  such that  $L(n) = m_0n$ , for some  $m_0 \in LUC(G)^*$ , then using a right cancellable element  $z \in C_0(G)^\perp$  - see Lemma 3.4 - we have that  $R(n) = nm_0$ :

$$R(n)z = nL(z) = nm_0z.$$

**Proposition 3.13.** *Suppose that  $G$  and  $H$  are locally compact groups and  $T : C_0(G)^\perp \rightarrow C_0(H)^\perp$  is an isometric isomorphism. Then for each  $x \in H$ , the isometric right multiplier  $R_x(m) = T^{-1}(T(m)\delta_x)$  is weak-star continuous.*

*Proof.* It is enough to show that  $R_x$  is weak-star continuous on the unit ball of  $C_0(G)^\perp$ . Suppose that  $n_i \xrightarrow{w^*} n$  in  $C_0(G)^\perp$ , with  $(n_i)$  bounded in norm by 1. Clearly  $(R_x(n_i))$  is also bounded by 1. Suppose that  $z \in G^*$ , is right cancellable in  $G^{luc}$ . We have that

$$\begin{aligned} R_x(n_i)\delta_z &= n_iL_x(\delta_z) \\ &\xrightarrow{w^*} nL_x(\delta_z) = R_x(n)\delta_z. \end{aligned}$$

By Lemma 3.4,  $R_x(n_i) \rightarrow R_x(n)$  weak-star in  $C_0(G)^\perp$ , as needed.  $\square$

Another related interesting question is whether there is a characterization for the multipliers on  $C_0(G)^\perp$ . Unlike,  $LUC(G)^*$ ,  $C_0(G)^\perp$  does not possess a unit or even an approximate identity. In fact, it is not hard to see that the annihilator of  $C_0(G)^\perp$  in  $LUC(G)^*$  is zero. Noting that  $C_0(G)^\perp$  is a closed ideal in  $LUC(G)^*$ , there is a natural embedding of  $LUC(G)^*$  into  $\mathcal{D}(C_0(G)^\perp)$ , the double centralizer algebra of  $C_0(G)^\perp$ . Does  $LUC(G)^* = \mathcal{D}(C_0(G)^\perp)$ ? Is there any nice characterization for  $\mathcal{D}(C_0(G)^\perp)$ ?

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